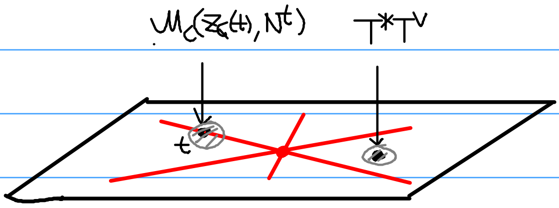
Addendum

$$H_*^{G_\mathcal{O}}(\mathcal{R}) \longleftarrow H^*_G(pt)$$

$$\mathcal{M}_C \longrightarrow \mathfrak{t}/W$$

$H_*^{T_\mathcal{O}}(\mathcal{R})_t \cong H_*^{T_\mathcal{O}}(\mathcal{R}^t)_t$ holds for any t

$\forall t \quad \mathcal{R}^t = \mathcal{R}_{(Z_G(t), N^t)}$ ∴ is the Coulomb branch for a "smaller" gauge theory



This picture is used to prove that $\mathcal{M}_C \cong$ (physical) candidate

motivation/conjecture

"Conjecture" We have 3d TQFT for the gauge theory (G, M)

X^3 : 3-mfd without ∂ $\rightsquigarrow$ $Z_{G,M}(X)$ "$\in$" $\mathbb{C}$ need to be corrected
oriented

Σ : 2-mfd without ∂ $\rightsquigarrow$ $Z_{G,M}(\Sigma)$: Hilbert space
oriented

$\partial X^3 = \Sigma \quad \rightsquigarrow \quad Z_{G,M}(X) \in Z_{G,M}(\Sigma)$

+ gluing axiom , $Z(-\Sigma) = Z(\Sigma)^*$ etc

Phys. def $\Rightarrow$ Gauge$(G, M) \cong \sigma$-model to $\mathcal{M}_C$,

$\therefore \quad Z_{G,M} = Z_{\sigma\text{-model}} =$ Rozansky-Witten theory

Fact $\quad \underset{\substack{\text{with} \\ \text{target } \mathcal{M}}}{Z_{\sigma\text{-model}}}(S^2) = \bigoplus_q H^q(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$

"Cor" 1 $\quad Z_{G,M}(S^2) = \mathbb{C}[\mathcal{M}_C] \quad$ as $\mathcal{M}_C$: affine variety

Rem. RW : it is "hazardous" to apply this definition to noncompact $\mathcal{M}$

"Cor" 2 $\quad Z_{G,M}(S^2 \times S^1) = \dim Z_{G,M}(S^2) = \dim \mathbb{C}[\mathcal{M}_C] = \infty$

Recall $S^1 \curvearrowright \mathcal{M}_C \quad \therefore \mathbb{C}[\mathcal{M}_C]$: S^1-module

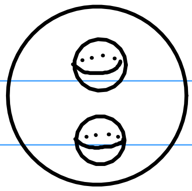
So $\quad \mathrm{ch}\, \mathbb{C}[\mathcal{M}_C] = \sum_m t^m \dim \underbrace{\mathbb{C}[\mathcal{M}_C]_m}_{\text{weight } m}$

could be well-defined

e.g. $\mathcal{M}_C = \mathbb{C}^2 = \mathrm{Spec}\, \mathbb{C}[x, y] \qquad x, y$: wt $= 1$

$\mathrm{ch}\, \mathbb{C}[\mathcal{M}_C] = 1 + 2t + 3t^2 + \cdots\cdots$

Fact $\quad \mathrm{Casson}(S^2 \times S^1) = -\frac{1}{12} = \zeta(-1) \qquad$ So match with

$X^3 =$  $\Rightarrow Z_{M,G}(X) \in \mathrm{Hom}(Z_{M,G}(S^2)^{\otimes 2}, Z_{M,G}(S^2))$

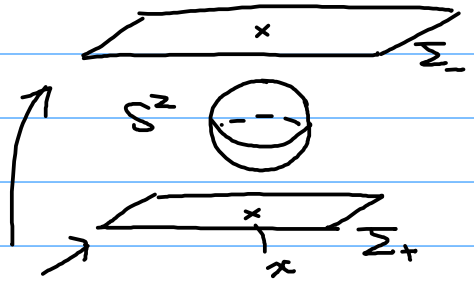
commutative multiplication

$\therefore \mathcal{M}_C = \mathrm{Spec}\,(Z_{M,G}(S^2), \text{mult} = Z_{M,G}(X))$

So enough to define

S^2 vs $D = \mathrm{Spec}\,\mathbb{C}[[z]]$

We need to break 3^d symmetry and consider $3 = 2+1$



$Z(X) \in \mathrm{Hom}(Z(\Sigma_+) \otimes Z(S^2), Z(\Sigma_-))$

look at only Σ_+, Σ_- $\quad$ G-b'dle $\mathcal{E}_+$ on Σ_+ is twisted by a b'dle over S^2 to get a b'dle $\mathcal{E}_-$ on Σ_-

$\rightarrow$ $\mathcal{E}_+, \mathcal{E}_-$ isomorphic outside x

section s is defined over S^2 $\rightarrow$ both sections of $\mathcal{E}_\pm$ simultaneously.

$\rightsquigarrow Z_{(G,M)}(S^2) = H_*^{G_\mathcal{O}}(\mathcal{E})$ $\quad$ at least if $M = N \oplus N^*$

Rem $\quad Z_{\text{RW with target } \mathcal{M}}(\Sigma_g) = H^*(\mathcal{M}, (\Lambda^* T_\mathcal{M})^{\otimes g})$

Compatible with conjecture on mixed Hodge pol of char. varieties

by Hausel-Rodriguez-Villegas

$(G, M) = (GL(n), \mathfrak{gl}(n)) \rightsquigarrow \mathcal{M} = \mathrm{Hilb}^n(\mathbb{C} \times \mathbb{C}^\times)$

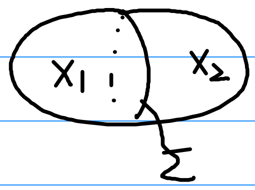
In conventional mathematical approach to $Z_{G,M}$, people use moduli spaces of nonlinear PDE on X or Σ

$\mathcal{M}_X$ = moduli space of sol. on X $\quad Z_{G,M}(X) = \#^{vir}\mathcal{M}_X$

$\mathcal{M}_\Sigma$ = 〃 Σ $\quad Z_{G,M}(\Sigma) = H^*(\mathcal{M}_\Sigma)$

$\partial X = \Sigma \to \mathcal{M}_X \xrightarrow{\text{bdry}} \mathcal{M}_\Sigma \quad [\mathcal{M}_X] \in H^*(\mathcal{M}_\Sigma)$

"lagrangian"

$X =$ 

$\#^{vir}\mathcal{M}_X = \langle [\mathcal{M}_{X_1}], [\mathcal{M}_{X_2}] \rangle$

This construction was worked out (partially)

if $(G = SL(2), M = 0)$, $(G = U(1), M = \mathbb{C} \oplus \mathbb{C}^*)$

instantanton Floer $\qquad$ Heegard Floer

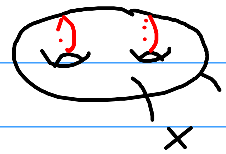
$\mathcal{M}_\Sigma$ = moduli of flat connections

$= \mathrm{Hom}(\pi_1(\Sigma), SU(2))/\text{conj}$

$\mathcal{M}_X = \mathrm{Hom}(\pi_1(X), SU(2))/\text{conj}.$

$\mathcal{M}_\Sigma$ = moduli of line b'dle + section $= S^g\Sigma_g$

$\mathcal{M}_X$ = image of attaching cycles



But $\Sigma = S^2 \quad \mathcal{M} \overset{SU(2)}{=} pt/SU(2)$, $\overset{U(1)}{=} pt \qquad H^*_{SU(2)}(pt) = \mathbb{C}/\mathbb{Z}_2$

not correct!

So need a correction!

<u>Conjecture</u> $\quad M$: general

$\mathbb{C}[\mathcal{M}_C] = H^*_c(\text{moduli stack of } (\mathcal{E}, s) \text{ s.t. } \mu_{\mathbb{C}}(s) = 0, \varphi_{\mathbb{C}S}(\mathbb{C}_{\mathfrak{g}_1}))^\vee$

G-b'dle on $S^2 = \mathbb{P}^1$ $\qquad$ holo. section of $(\mathcal{E} \times_G M) \otimes \mathcal{O}(-1)$

$\mathfrak{F}$ = the space of all fields $\qquad \Sigma$ = a cpt Riemann surface

$= \{ (P, \bar\partial + A, s) \}$

C^∞ G-b'dle $\quad$ partial connection on a G-b'dle $\quad$ C^∞-section of $(P \times_G M) \otimes K_\Sigma^{1/2}$

$$CS : \mathfrak{F} \longrightarrow \mathbb{C}$$

$$CS(A, \Phi) = \int_\Sigma \omega_{\mathbb{C}}(\underbrace{\bar\partial_A \Phi}_{K_\Sigma^{1/2} \otimes \Omega^{0,1} \otimes P \times_G M} \wedge \underbrace{\Phi}_{K_\Sigma^{1/2} \otimes P \times_G M})$$

$$\begin{cases} 0 = dCS(0, \dot\Phi) \quad \forall \dot\Phi & \Longleftrightarrow \quad \bar\partial_A \Phi = 0 \\ 0 = dCS(\dot A, 0) \quad \forall \dot A & \Longleftrightarrow \quad \mu_{\mathbb{C}}(\Phi) = 0 \end{cases}$$

So one hope to get $\quad \varphi_{CS}(\mathbb{C})$: sheaf on Crit CS

vanishing cycle fct

<u>Remark</u>

$M = N \oplus N^*$ $\quad$ one can "cut" N^*

$\circlearrowleft \mathbb{C}^\times$

- Behrend–Bryan–Szendroi
- MMNS
- Davison

$(\mathrm{crit}\, CS, \varphi_{CS}(\mathbb{C}_{\mathfrak{F}}))$

$\rightsquigarrow$ smaller moduli sp. & + constant coefficient